VIBRATIONS OF AIRCRAFT WING TYPE STRUCTURES

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Mr.S.B.Rao, who worked as a S.R.F. for a period, last year, had started the work on the development of thin walled beam element.

VIBRATIONS OF AIRCRAFT WINGTYPE STRUCTURES - I

Summary

A new method for the analysis of vibrations of swept box-beams, representative of low aspect ratio swept wings, has been presented. The root part of the wing is idealised using plane stress triangular elements, whereas the out board part is idealised using thin-walled beam elements. Proper matching of both the regions is achieved through the introduction of a special element called 'junction element'. The feasibility of the scheme is confirmed, through numerical experiments, on a simple example of a cantilevered rectangular unswept box beam.

1. Introduction

We mean by aircraft wing-type structures, hollow beams with single or multi-cell, closed or open cross-section. These are characterised by the feature that deformations in the plane of cross section are negligible. Conventional aircraft wings with spars, ribs ... etc., vertical or horizontal tails, control surfaces such as elevator, rudder, aileron are a few examples.

For the analysis of these structures classical theory of beams is inadequate. Whereas the use of plate or shell theories will be too complicated. Thus there has been a need to introduce a theory intermediate between the two extreme situations-beam theory on one hand and shell theory on the other. Based on the fact that the construction of wing structures (i.e. the presence of ribs) is such that the deformations in the plane of the cross-section are negligible and at the same time, the out of plane deformations are significant, the famous assumption of "closely spaced rigid diaphragm" or CSRD has been introduced. Making use of this assumption a theory has been developed in middle forties for static analysis of wings [1-6]. This theory has been referred to, in the literature, as thin-walled beam theory, or refined beam theory or tube theory. In this report this shall be referred to as thin-walled beam theory. This theory has proved to be very useful for the analysis of several stressing situations in wings. Utilizing this theory several data sheets have also been prepared Ob for direct use in the design offices of aircraft factories.

Work on vibration analysis of aircraft structures has started in early fifties. With increased use of short aspect ratio wings in aircraft, the need to use refined beam theories for their vibration analysis has increased. In middle sixties, the principal investigator and Joga Rao have developed a generalised theory for vibration analysis of cylindrical wings [10, 11] . Later, it was generalised to consider general wing configuration [12, 13] . Since middle sixties, the developments in the finite element theory began influencing the approach for the analysis of wings. In view of the inherent advantages of the finite element method for high speed numerical computation and its suitability for complex structures, such as wings, to-day, the position appears to be that the finite element theory completely took over the analysis of wings. The conventional finite element approach for wing analysis consists of idealising the wing as an assemblage of a single type two dimensional elements, such as plane stress triangular elements or triangular plate bending elements, for panels and line elements for stiffeners (see Fig.1).

An examination of the behaviour of a typical wing, such as a short aspect ratio swept wing shown in Figure 4, indicates that the out board part of the wing behaves like a thin-walled beam, whereas near the root the stressing pattern is very complicated. Therefore, it will be advantageous, in the finite element scheme, to use elements based on thin-walled beam theory in the out-board part of the wing and use conventional finite element idealisation near root portion.

Adaptation of such an approach involves two types of nodes. One type, with a certain degrees of nodal freedoms, in the root portion, and, a different type, with a different set of nodal freedoms, in the out board portion. This presents a problem of matching of both these regions. This is overcome, here, by the introduction of an interface region called junction element. This is essentially a large special element, with both the types of nodes on appropriate boundaries of the elements, facilitating smooth matching of both the regions.

In this report, the wing is idealised as a swept box of trepezoidal cross-section. Following the above mentioned approach, a finite element scheme has been formulated for the vibration analysis, and computer program has been preparated in FORTRAN-IV.

It is proposed to test and confirm the scheme step by step. As a first step, a rectangular unswept cantilevered box-beam, which is amenable for analysis by thin-walled beam theory itself, has been chosen for numerical work. Natural frequencies obtained by the 'junction element' method are compared with those obtained using thin-walled beam elements only. The agreement between them is found to be good, thus confirming the feasibility of the junction element scheme.

Notation

A	: Area of cross section
E	: Young's modulus
G	: Rigidity modulus
I	: Lowest moment of inertia of cross section
L	: Length of the wing
S	: Strain energy
s	: Circumferential co-ordinate
t	: Wall thickness of the wing
T	: Kinetic energy
u,v,w	: Displacement of any point in the wing in local co-ordinates
U,V,W	: Displacement of any point in the wing in global co-ordinates
U_o, V_o, W_o	: Displacements of any point on the centroidal axis of the wing
$\bar{w}$	: Warping function
α', β'	: Taper ratios
ϵ_x, ϵ_y	: Direct strains
ϵ_{xy}	: Shearing strains
θ	: Rotation of the cross-section
λ_o	: Non-dimensional frequency parameter (Fundamental)

ρ	: Mass density
ν	: Poisson's ratio
ξ	: Non-dimensional length along the element
σ_x, σ_y	: Direct stresses
σ_{xy}	: Shearing stress

<u>Matrices</u>

$\lceil D \rfloor$	: Elasticity matrix
$[L]$	: Matrix of differential operators
$[N]$	: Matrix of shape functions
$[k_\delta]$	: Element stiffness matrix with reference to nodal displacement vector, δ
$[m_\delta]$	: Element mass matrix with reference to nodal displacement vector, δ
$[a]$	: Displacement transformation matrix connecting local and global co-ordinates
$\{\Delta\}$	: Vector of displacements at any point in the wing in local co-ordinates
$\{\delta\}$	: Vector of nodal displacements in local co-ordinates.
$\{\epsilon\}$	: Vector of strains
$\{\eta\}$	: Vector of displacements at any point in the wing in global co-ordinates

$[\lambda]$	: Matrix of direction consines
$\{\sigma\}$	: Vector of stresses
$\lceil\ \ \rfloor$	: Denotes diogonal matrix
$\{\ \}$	: Denotes column matrix
$,s$	: Denotes differentiation with reference to s
(R), (J), (O)	: These super-scripts represent, respectively, the Root portion, the Junction portion and Out board portion.

2. Basic elements

In the finite element idealisation of the swept boxes, three types of elements are used.

1. A triangular panel element.
2. A thin-walled box beam element.
3. A 'junction element'.

These are presented below:

2.1. Triangular panel element

One of the most versatile elements available in literature is the well-known constant stress triangle(18). This element will be used here for the idealisation of the root part of the wings. Derivation of the relevant element matrices, although available in literature, is reproduced here for the sake of completeness.

2.1.1. Theoretical basis

The panel is considered to be in a state of plane stress (see Figure 2a). The state of stress $\left\{\sigma^{(R)}\right\}$, strain $\left\{\epsilon^{(R)}\right\}$ and displacement $\left\{\Delta^{(R)}\right\}$ at any point in the panel can be described by

$$\left\{\sigma^{(R)}\right\} = \left\{\sigma_x , \sigma_y , \sigma_{xy}\right\} \qquad (1)$$

$$^{*}\left\{\epsilon^{(R)}\right\} = \left\{\epsilon_x,\ \epsilon_y,\ \epsilon_{xy}\right\} \tag{2}$$

$$\left\{\Delta^{(R)}\right\} = \left\{u,\ v\right\} \tag{3}$$

The material is assumed to obey Hooke's law and the deformations are considered to be small. The stress strain and the strain-displacement relations are

$$\left\{\sigma^{(R)}\right\} = \left\lceil D^{(R)} \right\rfloor \left\{\epsilon^{(R)}\right\} \tag{4}$$

$$\left\{\epsilon^{(R)}\right\} = \left[L^{(R)}\right] \left\{\Delta^{(R)}\right\} \tag{5}$$

where

$$\left\lceil D^{(R)} \right\rfloor = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 1 & \frac{1-\nu}{2} \end{bmatrix} \tag{6}$$

$$\left[L^{(R)}\right] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{7}$$

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* In plane stress problems $\epsilon_z \neq 0$ but is determinate in terms of ϵ_x and ϵ_y as, $\epsilon_z = \nu(\epsilon_x + \epsilon_y)$ and hence not included in the strain vector.

2.1.2. Element stiffness and mass matrices

A typical triangular element along with the coordinate system is shown in Figure 2.(b). The displacement shape functions in the element, defined by the equation

$$\left\{\Delta^{(R)}\right\} = \left[N^{(R)}\right] \left\{\delta^{(R)}\right\} \tag{8}$$

are

$$\left[N^{(R)}\right] = \frac{1}{2A_{123}} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{9}$$

where

$$N_1 = y_{32}(x-x_2) - x_{32}(y-y_2)$$

$$N_2 = - y_{31}(x-x_3) + x_{31}(y-y_3)$$

$$N_3 = y_{21}(x-x_1) - x_{21}(y-y_1)$$

$$2A_{123} = (x_{32}\, y_{21} - x_{21}\, y_{32})$$

$$x_{ij} = x_i - x_j \,, \; y_{ij} = y_i - y_j \tag{10}$$

and $\left\{\delta^{(R)}\right\}$ is the vector element displacements given by

$$\left\{\delta^{(R)}\right\} = \left\{u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3\right\} \tag{11}$$

Following the standard procedure, the stiffness $\left[k_{\delta}^{(R)}\right]$ and ss $\left[m_{\delta}^{(R)}\right]$ matrices with respect to $\left\{\delta^{(R)}\right\}$ can be determined

$$\left[k_{\delta}^{(R)}\right] = \left[k_{\delta n}^{(R)}\right] + \left[k_{\delta s}^{(R)}\right] \tag{12}$$

$${}^{R)}_{m} = \frac{Et}{4A_{123}(1-\nu^2)} \begin{bmatrix} y_{32}^2 & & & & \text{Symmetric} & \\ -\nu y_{32}\,x_{32} & x_{32}^2 & & & & \\ -y_{32}\,y_{31} & \nu x_{32}\,y_{31} & y_{31}^2 & & & \\ \nu y_{32}\,x_{31} & -x_{32}\,x_{31} & -\nu y_{31}\,x_{31} & x_{31}^2 & & \\ y_{32}\,y_{21} & -\nu x_{32}\,y_{21} & -y_{31}\,y_{21} & \nu x_{31} y_{21} & y_{21}^2 & \\ -\nu y_{32}\,x_{21} & x_{32}\,x_{21} & \nu y_{31}\,x_{21} & -x_{31}x_{21} & -\nu y_{21}\,x_{21} & x_{21}^2 \end{bmatrix} \tag{13a}$$

$$\left[k^{(R)}_{\delta s} \right] = \frac{Et}{\delta A_{123}(1+\nu)} \begin{bmatrix} x_{32}^2 & & & & & \\ -x_{32}y_{32} & y_{32}^2 & & & & \\ -x_{32}x_{31} & y_{32}x_{31} & x_{31}^2 & & \text{Symmetric} & \\ x_{32}y_{31} & -x_{32}y_{31} & -x_{31}y_{31} & y_{31}^2 & & \\ x_{32}x_{21} & -y_{32}x_{21} & -x_{31}x_{21} & y_{31}x_{21} & x_{21}^2 & \\ -x_{32}y_{21} & y_{32}y_{21} & x_{31}y_{21} & -y_{31}y_{21} & -x_{21}y_{21} & y_{21}^2 \end{bmatrix} \tag{13b}$$

where (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are coordinates of the points p, q, r, in the coordinate system p_{xy} as

$$x_1 = y_1 = x_2 = 0$$

$$y_2 = d_{pq}$$

$$x_3 = d_{pt}$$

$$y_3 = d_{tr}$$

d_{pq}--are defined in equations (17), (19) and (20).

and

$$\left[m_{\delta}^{(R)}\right] = \frac{\mu A_{123} t}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \tag{14}$$

For assembling the element matrices, it will be convenient to convert element matrices with respect to the datum coordinates $\eta^{\cdot(R)}$ defined as (see Figure 2)

$$\left\{\eta^{(R)}\right\} = \left\{U_1\ U_2\ U_3\ \text{---}\ U_9\right\} \tag{15}$$

where U_1, U_2, --- U_9 are displacements of vertices in datum coordinates. The relationship between $\left\{\eta^{(R)}\right\}$ and $\left\{\delta^{(R)}\right\}$ systems of coordinates can be worked out as follows:

In Figure 2(b) OXYZ system forms the reference or the global cartesian system of coordinates. With reference to this the vertices of the triangle pqr are

$$\begin{aligned} p &= p(x_p,\ y_p,\ z_p) \\ q &= q(x_q,\ y_q,\ z_q) \\ r &= r(x_r,\ y_r,\ z_r) \end{aligned} \tag{16}$$

pxy consistitutes the local coordinate system. px axis coincides with the side pq and py axis is taken normal to pq at p.

The local coordinates of the vertices of the triangle are calculated using their respective global coordinates, the direction cosines of the edge pq and direction cosines of the normal to pq. i.e. the direction tr.

The direction cosines of pq are:

$$l_{pq} = \frac{x_{pq}}{d_{pq}}$$

$$m_{pq} = \frac{y_{pq}}{d_{pq}}$$

$$n_{pq} = \frac{z_{pq}}{d_{pq}}$$

and $$d_{pq} = (x_{qp}^2 + y_{qp}^2 + z_{qp}^2)^{1/2} \qquad (17)$$

The direction cosines of tr can be shown to be [18]

$$l_{tr} = \frac{x_{rp} - l_{pq}\, d_{pt}}{d_{tr}}$$

$$m_{tr} = \frac{y_{rp} - m_{pq}\, d_{pt}}{d_{tr}} \qquad (18)$$

$$n_{tr} = \frac{z_{rp} - n_{pq}\, d_{pt}}{d_{tr}}$$

where

$$d_{pt} = (l_{pq}\ x_{rp} + m_{pq}\ y_{rp} + n_{pq}\ z_{rp})^{1/2} \tag{19}$$

$$d_{tr} = (x_{rp}^2 + y_{rp}^2 + z_{rp}^2 - d_{pt}^2)^{1/2} \tag{20}$$

Using above, the relationship between $\{\delta^{(R)}\}$ and $\{\eta^{(R)}\}$ systems may be written as

$$\{\delta^{(R)}\} = [\lambda]\ \{\eta^{(R)}\} \tag{21}$$

where $$[\lambda] = \begin{bmatrix} [\lambda_{tr}] & 0 & 0 \\ [\lambda_{pq}] & 0 & 0 \\ 0 & [\lambda_{tr}] & 0 \\ 0 & [\lambda_{pq}] & 0 \\ 0 & 0 & [\lambda_{tr}] \\ 0 & 0 & [\lambda_{pq}] \end{bmatrix} \tag{22}$$

with $$[\lambda_{tr}] = [l_{tr}\quad m_{tr}\quad n_{tr}] \tag{23}$$

$$[\lambda_{pq}] = [l_{pq}\quad m_{pq}\quad n_{pq}] \tag{24}$$

and '0' indicates null row matrix of order 3.

Using $[\lambda]$, the stiffness $\left[k_{\eta}^{(R)}\right]$ and mass $\left[m_{\eta}^{(R)}\right]$ matrices with reference to $\left\{\eta^{(R)}\right\}$ system of element coordinates become

$$\left[k_{\eta}^{(R)}\right] = [\lambda]^{T} \left[k_{\delta}^{(R)}\right] [\lambda] \tag{26}$$

$$\left[m_{\eta}^{(R)}\right] = [\lambda]^{T} \left[m_{\delta}^{(R)}\right] [\lambda] \tag{26}$$

2.2. Thin-walled box beam element

2.2.1. Theoretical basis

A thin walled beam is a hollow beam, with one or more cells, closed or open. The length of the beam is much larger than the cross-sectional dimensions. The walls of the beams are thin. The out board part of the wing, for example the region EFGH in Figure 4a, can be considered as a thin-walled box-beam. For vibration analysis of such structural components, a general theory has been developed in Ref. [7-13]. The essential parts of this theory as applied to a single cell box beam, which are used later in the development of the Box-beam element, are repeated here for the sake of completeness.

The principal assumption in this theory is that the cross-sectional shape is maintained through a system of closely spaced rigid and massless diaphragms. With this assumption the state of stress $\{\sigma^{(0)}\}$ strain $\{\epsilon^{(0)}\}$ and displacement field $\{\bar{\Delta}^{(0)}\}$ at any point in the beam are described by

$$\{\sigma^{(0)}\} = \{\sigma_Z \quad \sigma_{Zs}\} \tag{27}$$

$$\{\epsilon^{(0)}\} = \{\epsilon_Z \quad \epsilon_{Zs}\} \tag{28}$$

$$\{\bar{\Delta}\} = \{U \quad V \quad W\} \tag{29}$$

The stress strain and strain-displacement relationships

$$\{\sigma^{(0)}\} = \lceil D^{0} \rfloor \{\epsilon^{(0)}\}$$

$$\{\epsilon^{(0)}\} = [\bar{L}]\{\bar{\Delta}\}$$

$$\lceil D^{(0)} \rfloor = \lceil E, G \rfloor$$

where $$[\bar{L}] = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial z} \\ x,s\frac{d}{dz} & y,s\frac{d}{dz} & p\frac{d}{dz} & \frac{\partial}{\partial s} \end{bmatrix}$$

where p is the perpendicular from origin to tan
To the first order approximation, 'W' can be ap
with reference to the centroidal axis as

$$W = - x\, U_{o,z} - y\, V_{o,z} - \bar{w}\, \theta_{,z} - x\phi_x - $$

are centroidal displacements

of the cross-section and $\bar{w}$ is the warping function associated with the torsion of the cross-section, ϕ_x, ϕ_y and ϕ_θ are unknown functions of z. In view of this choice, the appropriate vector, describing the displacement field becomes

(35)

$$\left\{\Delta^{(0)}\right\} = \left\{U_o,\ V_o,\ \theta,\ \phi_x,\ \phi_y,\ \phi_\theta\right\}$$

and the strain-displacement relation becomes

(36)

$$\left\{\epsilon^{(0)}\right\} = \left[L^{(0)}\right] \left\{\Delta^{(0)}\right\}$$

where

$$\left[L^{(0)}\right] = \begin{bmatrix} -x\frac{d^2}{dz^2} & -y\frac{d^2}{dz^2} & -\bar{w}\frac{d^2}{dz^2} & -x\frac{d}{dz} & -y\frac{d}{dz} & -\bar{w}\frac{d}{dz} \\ 0 & 0 & (p-\frac{d\bar{w}}{ds})\frac{d}{dz} & -x_{,s} & -y_{,s} & -\bar{w}_{,s} \end{bmatrix}$$

(3

$\left\{\bar{\Delta}\right\}$ and $\left\{\Delta^{(0)}\right\}$ are related as

$$= a_1 \quad (0)$$

where

$$\left[a_1\right] = \begin{bmatrix} 1 & 0 & -y & 0 & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ -x\frac{d}{dz} & -y\frac{d}{dz} & -\bar{w}\frac{d}{dz} & -x & -y & -\bar{w} \end{bmatrix}$$

2.2.2. Element stiffness and mass matrices

A typical element, along with the element nodal displacements, is shown in Figure 3. The vector of element nodal displacements is taken as

$$\{\delta^{(0)}\} = \{U_{o1}\ U_{o1,z}\ V_{o1}\ V_{o1,z}\ \theta_1\ \theta_{1,z}\ \ \phi_{x1}\ \phi_{y1}\ \phi_{o1} \mid U_{o2}, ---- \phi_{o2}\} \tag{40}$$

The displacement field is chosen as

$$\{\Delta^{(0)}\} = [N^{(0)}]\ \{\delta^{(0)}\} \tag{41}$$

where

$$[N^{(0)}] = \begin{bmatrix} H_1 & H_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_3 & H_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_1 & H_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_3 & H_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_1 & H_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_3 & H_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_2 \end{bmatrix} \tag{42}$$

where $H_1 = 1 - 3\xi^2 + 2\xi^3$

$H_2 = \xi - 2\xi^2 + \xi^3$

$H_3 = 3\xi^2 - 2\xi^3$

and

$$F_1 = 1 - \xi$$

$$F_2 = \xi \tag{43}$$

For convenience the differential operator matrix $L^{(0)}$ is written as

$$\left[L^{(0)}\right] = \left[a_2\right] \lceil d_1 \rfloor + \left[a_3\right] \lceil d_2 \rfloor \tag{44}$$

where

$$\left[a_2\right] = \begin{bmatrix} -x & -y & -\bar{w} & -x & -y & -\bar{w} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{45}$$

$$\lceil d_1 \rfloor = \lceil \,{,zz} \quad {,zz} \quad {,zz} \quad {,z} \quad {,z} \quad {,z} \,\rfloor \tag{46}$$

$$\left[a_3\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (p - \bar{w}_{,s}) & -x_{,s} & -y_{,s} & -\bar{w}_{,s} \end{bmatrix} \tag{47}$$

$$\lceil d_2 \rfloor = \lceil\, 0 \quad 0 \quad {,z} \quad 1 \quad 1 \quad 1 \,\rfloor \tag{48}$$

Using equation (42) we get

$$\left\{\epsilon^{(0)}\right\} = \left[\left[a_2\right]\left[B_{\delta_1}\right] + \left[a_3\right]\left[B_{\delta_3}\right]\right]\left\{\delta^{(0)}\right\} \tag{49}$$

where

$$\left[B_{\delta_1}\right] = \lceil d_1 \rfloor \left[N^{(0)}\right] \tag{50}$$

$$\left[B_{\delta_2}\right] = \lceil d_2 \rfloor \left[N^{(0)}\right] \tag{51}$$

Utilizing equation (49) the strain energy in the element becomes

$$S = 1/2 \int_V \{\epsilon^{(0)}\}^T \lceil D^{(0)} \rfloor \{\epsilon^{(0)}\} \, dV \tag{52}$$

or

$$S = 1/2 \{\delta^{(0)}\}^T \left[k^{(0)}\right] \{\delta^{(0)}\} \tag{53}$$

$\left[k^{(0)}\right]$ is the element stiffness matrix and is obtained as

$$\left[k^{(0)}\right] = \left[k_1\right] + \left[k_2\right] \tag{54}$$

$$\left[k_1\right] = \int_0^1 E \left[B_{\delta_1}\right]^T \left[C_1\right] \left[B_{\delta_1}\right] d\xi \tag{55}$$

$$\left[k_2\right] = \int_0^1 G \left[B_{\delta_2}\right]^T \left[C_2\right] \left[B_{\delta_2}\right] d\xi \tag{56}$$

$\left[C_1\right]$ and $\left[C_2\right]$ are matrices of cross-sectional constants, obtained as

$$\left[C_1\right] = \oint_s \left[a_2\right]^T \left[a_2\right] t ds \tag{57}$$

$$\left[C_2\right] = \oint_s \left[a_3\right]^T \left[a_3\right] t ds \tag{58}$$

These matrices are given in Appendix - A in terms of the
... of the cross-section.

an equivalent single node of thin walled beam type. Thus, junction element on the side connecting the region (R) will have nodes of the type 'R' and on the side connecting the region (O) will have nodes of the type 'O' thus facilitating the connection of both the regions. Details of development of this element are given below with reference to a specific finite element idealisation.

2.3.1. 'Junction Element' stiffness and mass matrices

Figures 4(b) and 4(c) show the idealisation of the junction for a specific finite element layout. Considering the domain 'J' as an assembly of triangular elements it has 28 nodes alltogether, 14 on the root side and 14 on the outboard side. Each node has three degrees of freedom. The assembled stiffness matrix $\left[k^{(1)}\right]$ for the domain 'J' will be of the size (84 x 84). $\left\{\delta_1\right\}$, the vector global displacements in the domain 'J' is written as

$$\left\{\delta_1\right\} = \left\{ \left\{\delta_I\right\} \,\vdots\, \left\{\delta_{II}\right\} \right\} \tag{71}$$

where

$$\left\{\delta_I\right\} = \left\{U_1\ V_1\ W_1,\ U_2\ V_2\ W_2\ ---,\ U_{14}\ V_{14}\ W_{14}\right\} \tag{72}$$

$$\left\{\delta_{II}\right\} = \left\{U_{15}\ V_{15}\ W_{15},\ U_{16}\ V_{16}\ W_{16}\ ---,\ U_{28}\ V_{28}\ W_{28}\right\} \tag{73}$$

The nodal displacements associated with the node 'A' of the domain 'O' (see Figures 4(b) - 4(e)

$$\left\{ \delta^{(A)} \right\} = \left\{ u_A \; u_{A,z} \; v_A \; v_{A,z} \; \theta_A \; \theta_{A,z} \; \phi_{xA} \; \phi_{yA} \; \phi_{\theta A} \right\} \tag{74}$$

Since the nodes 15 to 24 in the domain 'J' join the node A in domain 0, there is a relation-ship between $\{\delta_{II}\}$ and $\{\delta^{(A)}\}$, and this may be written as

$$\left\{\delta_{II}\right\} = \begin{bmatrix} U_{15} \\ V_{15} \\ W_{15} \\ \vdots \\ U_{28} \\ V_{28} \end{bmatrix} = \begin{bmatrix} \left[F(x_{15}, y_{15})\right] \\ \hline \vdots \\ \left[F(x_{28}, y_{28})\right] \end{bmatrix} \begin{bmatrix} u_A \\ u_{A,z} \\ \vdots \\ \phi_{yA} \\ \phi_{\theta A} \end{bmatrix} \tag{75}$$

here $\left[F(x_i, y_i)\right]$ is the relationship between the nodal isplacements of the i^{th} node in domain 'J', and that of the ode in domain 'O'.

sing equation (38) this can be written as

$$\left[F(x_i, y_i)\right] = \left[C\right]\left\{\delta_A\right\} \tag{76}$$

here

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & -y_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x_i & 0 & 0 & 0 & 0 \\ 0 & -x_i & 0 & -y_i & 0 & \bar{w}(x_i,y_i) & -x_i & -y_i & -\bar{w}(x_i,y_i) \end{bmatrix}$$

(77)

Thus $\{\rho_1\}$ may be written as

$$\{\rho_1\} = \left\{\{\rho_I\} \,\vdots\, \{\rho_{II}\}\right\} = \left[a^{(J)}\right]\left\{\{\rho_I\} \,\vdots\, \{\rho_A\}\right\} = \left[a^{(J)}\right]\left\{\rho^{(J)}\right\} \tag{78}$$

where

$$\left[a^{(J)}\right] = \left[\begin{array}{c|c} [I] & [0] \\ \hline [0] & [C] \end{array}\right] \tag{79}$$

with $[I]$ as the *unit matrix*

The stiffness matrix of the domain 'J' with respect to displacement vector $\left\{\rho^{(J)}\right\}$ becomes

$$\left[k^{(J)}\right]^{*} = \left[a^{(J)}\right]^{T} \left[k^{(J)}\right] \left[a^{(J)}\right] \tag{80}$$

Similarly the assembled mass matrix with respect to $\rho^{(J)}$ may be written as

$$\left[M^{(J)}\right]^{*} = \left[a^{(J)}\right]^{T} \left[M^{(1)}\right] \left[a^{(J)}\right] \tag{81}$$

It may be noted here $\left[k^{(J)}\right]$ and $\left[M^{(J)}\right]$ are singular with respect to the degrees of freedom u'_A, $u'_{\bar{A}}$, $-\theta_A$ in ρ^A and these are to be eliminated in addition to the ... rigid body modes to make it non-singular. However, when all the three domains are assembled, these degrees of freedom offer no ...

3.0 Formulation of the governing equations

A typical finite element idealisation of a wing is shown in Figure 4. $\left[M^{(R)}\right]$ and $\left[K^{(R)}\right]$ represent the assembled mass and stiffness matrices in the root region and can be written as

$$\left[M^{(R)}\right] = \left[a^{(R)}\right]^T \left[\diagdown m^{(R)} \diagdown\right] \left[a^{(R)}\right] \tag{82}$$

$$\left[K^{(R)}\right] = \left[a^{(R)}\right]^T \left[\diagdown k^{(R)} \diagdown\right] \left[a^{(R)}\right] \tag{83}$$

where $\left[a^{(R)}\right]$ is the displacement transformation matrix relating local and global coordinates in the region 'R'. Similarly the assembled mass and stiffness matrices in the outboard region are written as

$$\left[M^{(0)}\right] = \left[a^{(0)}\right]^T \left[\diagdown m^{(0)} \diagdown\right] \left[a^{(0)}\right] \tag{84}$$

and

$$\left[K^{(0)}\right] = \left[a^{(0)}\right]^T \left[\diagdown k^{(0)} \diagdown\right] \left[a^{(0)}\right] \tag{85}$$

where $\left[a^{(0)}\right]$ is the displacement transformation matrix relating local and global coordinates in the region '0'.

The assembled matrices of the whole wing may be obtained as

$$[K] = [a]^T \; [K] \; [a] \tag{86}$$

$$[M] = [a]^T \; [M] \; [a] \tag{87}$$

where

$$\left[K\right] = \begin{bmatrix} \left[K^{(R)}\right] & & \\ & \left[K^{(J)}\right] & \\ & & \left[K^{(O)}\right] \end{bmatrix} \tag{88}$$

$$\left[M\right] = \begin{bmatrix} \left[M^{(R)}\right] & & \\ & \left[M^{(J)}\right] & \\ & & \left[M^{(O)}\right] \end{bmatrix} \tag{89}$$

and a is the displacement transformation matrix defined as

$$\begin{bmatrix} \left\{\varrho^{(R)}\right\} \\ \left\{\varrho^{(J)}\right\} \\ \left\{\varrho^{(O)}\right\} \end{bmatrix} = \left[a\right]\left\{\varrho\right\} \tag{90}$$

where $\left\{\varrho\right\}$ is the vector of global coordinates for whole wing. The kinetic and strain energies for the whole wing are of the form

$$S = 1/2 \left\{\varrho\right\}^T \left[K\right]\left\{\varrho\right\} \tag{91}$$

$$T = 1/2 \ \omega^2 \left\{\varrho\right\}^T \left[M\right]\left\{\varrho\right\} \tag{92}$$

Using the stationary property of Lagrangion, (T-S), we get the governing equation for natural oscillations as

$$\left[K\right]\left\{\varrho\right\} - \omega^2 \left[M\right]\left\{\varrho\right\} = 0 \tag{93}$$

from which eigenv

One of the important tasks in a finite element solution is the development of a general purpose computer program. A program was written in FORTRAN-IV to compute the natural frequencies and mode shapes of a swept and tapered box-beam using plane stress triangular elements in the root portion, thin walled beam elements on the out board portion, the two regions appropriately connected by 'Junction element'. A simple flow chart is shown in Figure 5.

4. Results and Discussion

A thin walled cantilevered rectangular beam with the dimensions

Length $L = 142''$

Breadth $B = 48''$

Depth $D = 12''$

Wall thickness $t = 0.1''$

and with the material properties

Young's modulus $E = 2.9 \times 10^7$ psi.

Regidity modulus $G = 10^7$ psi.

was considered for numerical studies. The finite element subdivision used in the analysis is shown in the figure 6. The idealisation gives the structure 198 active degrees of freedom. The resulting eigenvalue problem was solved using an iteration technique.

Table 1 shows the fundamental frequency obtained using 15 thin-walled beam elements. Results by the classical slender beam theory are shown in the paranthesis. The box beam element developed here includes the effect of transverse shear and rotary inertia, whereas, these effects are not included in the classical slender beam theory. In the problem considered L/D ratio is about 11.8, hence, these effects are significant. The deviation between the results of the slender beam theory and the present analysis is an indication of the extent of influence of the latter effects. Table 2 presents the convergence trend.

Same problem is also investigated using the 'junction element' method. Near the fixed end 96 triangular elements are used. In the outboard portion 5 thin walled beam elements in one study and 8 elements in a second study, are used. Both the regions are connected through a junction element. The finite element idealisation of the thin-walled beam is shown in Figure 6. Table 3 presents the results. Since the problem considered is a straight box-beam the results by thin-walled beam elements can be considered as reasonably accurate, and the results by the Junction element agree well with these results.

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Table 1. Fundamental Frequency Parameters - Using Thin-walled Beam Elements.

$$\lambda_o = \frac{\omega_o^2 \; L^4 \; \mu A}{EI}$$

Taper Ratio α'	Taper Ratio β'	Fundamental Frequency parameter λ_o
0	0	8.43* (12.3627)
0.1	0	8.63+
0	0.1	10.92+
0.1	0.1	12.76+

* 11 Element Solution.

Values in the parantheses from elementary theory of bending vibrations.

\+ 15 element solution.

$$\alpha' = \frac{B_R - B_T}{B_R}$$

$$\beta' = \frac{D_R - D_T}{D_R}$$

R : Root dimension

T : Tip dimension

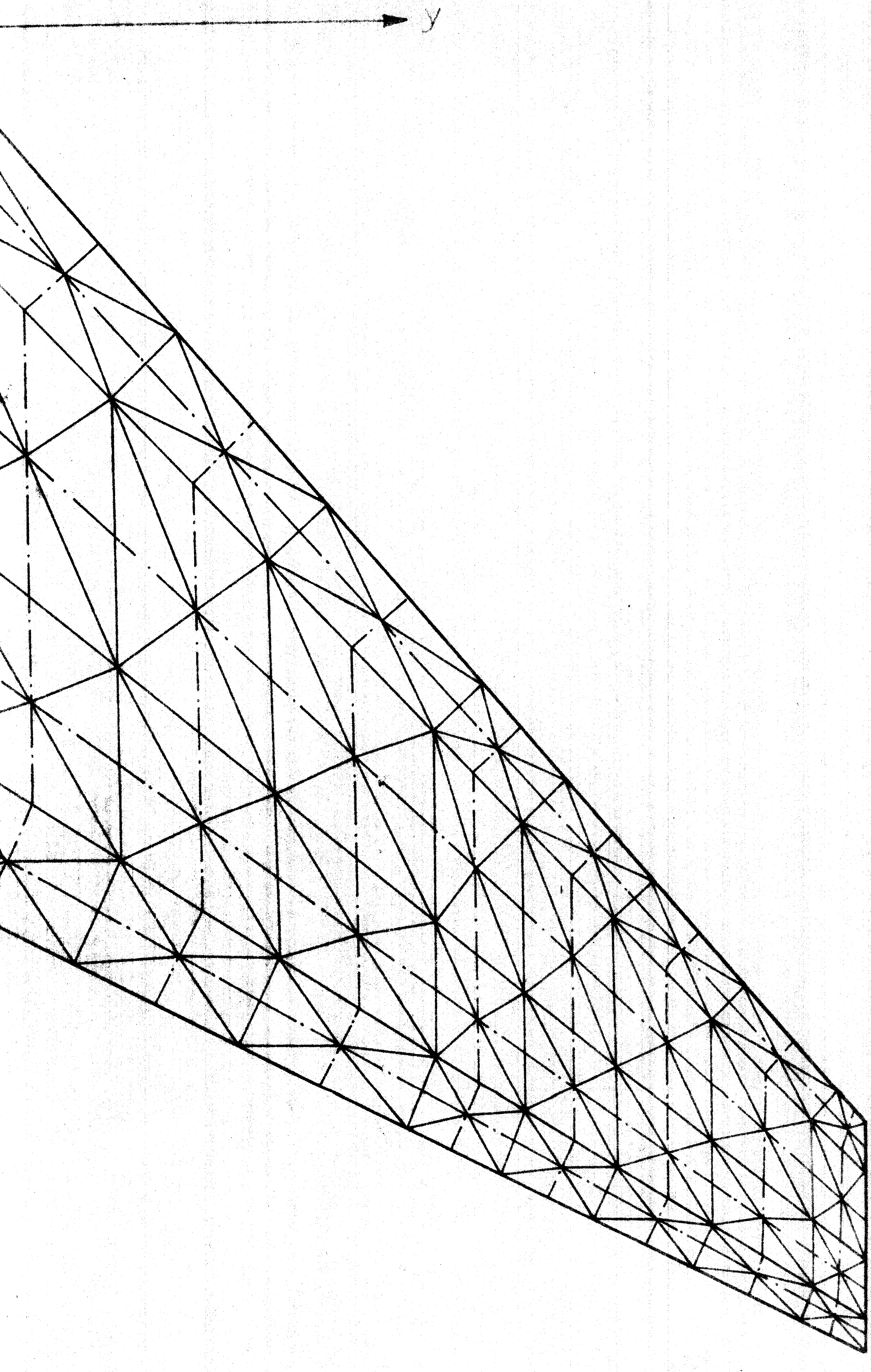
Table 2. Convergence using thin walled beam elements

No. of Elements	Size of dynamical Matrix	Fundamental Frequency parameter λ_o
5	45	145.66
6	54	78.72
7	63	43.92
8	72	26.92
9	81	17.6
10	90	11.94
11	99	8.43

Table 3. Fundamental Frequency Parameter by the 'Junction Element' Method.

Sl.No.	Idealisation	Fundamental Frequency parameter λ_o
1	96 Triangular Elements +1 Junction Element +5 Thin-walled beam elements	7.24 (8.43)*

* Using 11 thin walled beam elements.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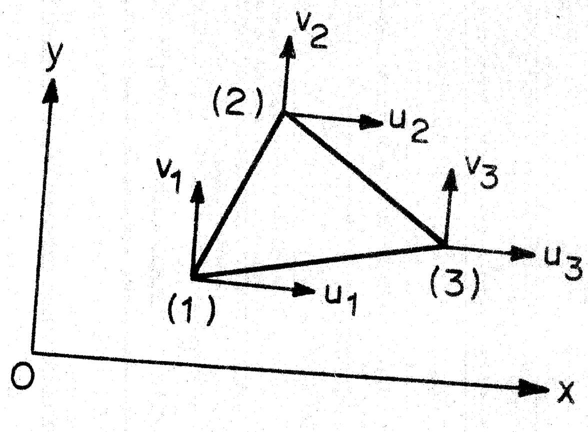
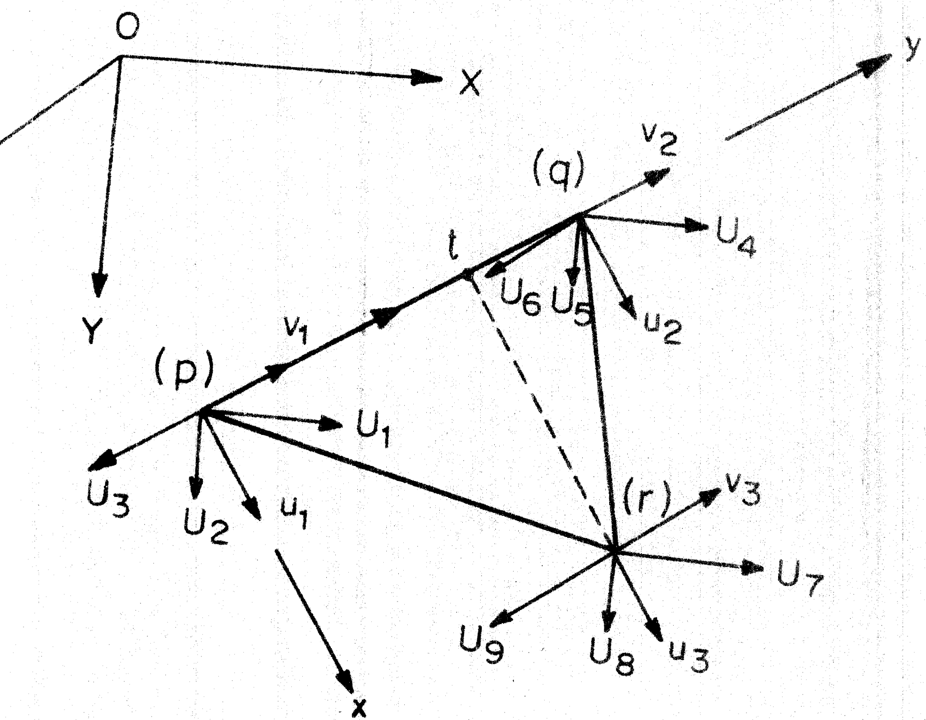


FIG. 2a : A TRIANGULAR PLATE UNDER PLANE STRESS .



O,X,Y,Z = Datum co-ordinate system

P x y = Local co-ordinate system

: TRIANGULAR PLATE DISPLACEMENTS IN LOCAL AND DATUM CO-ORDINATES .

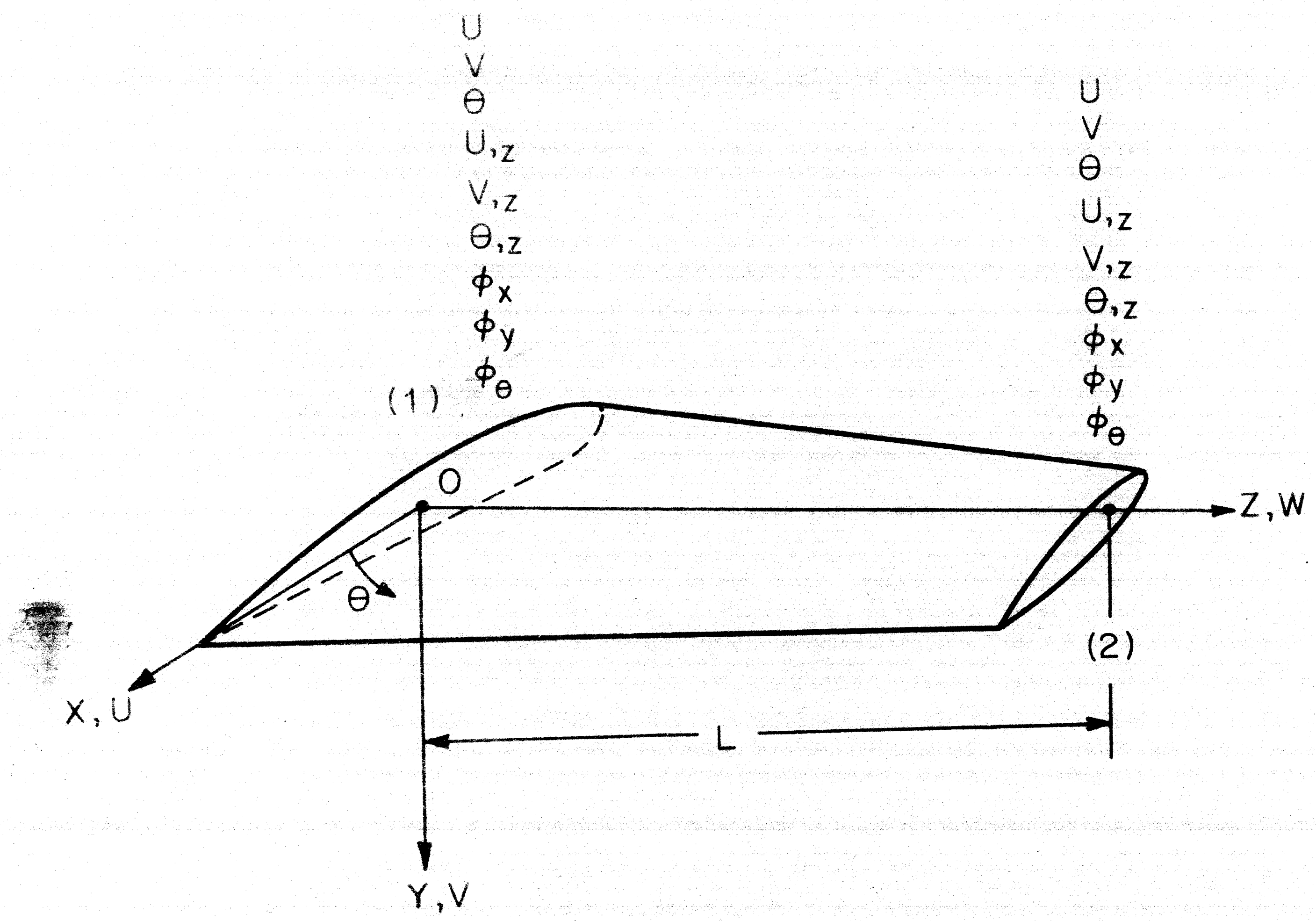
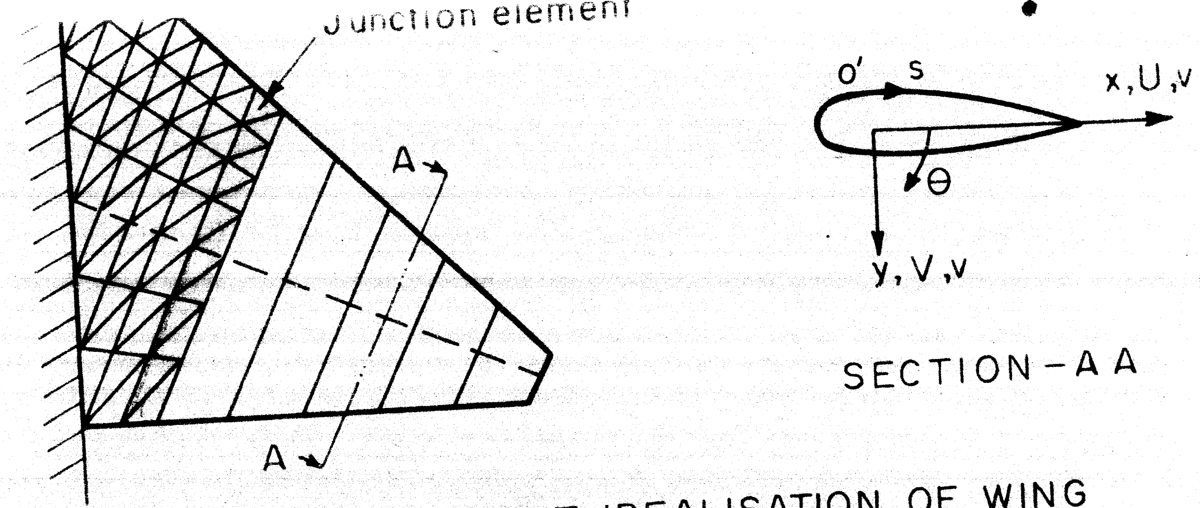
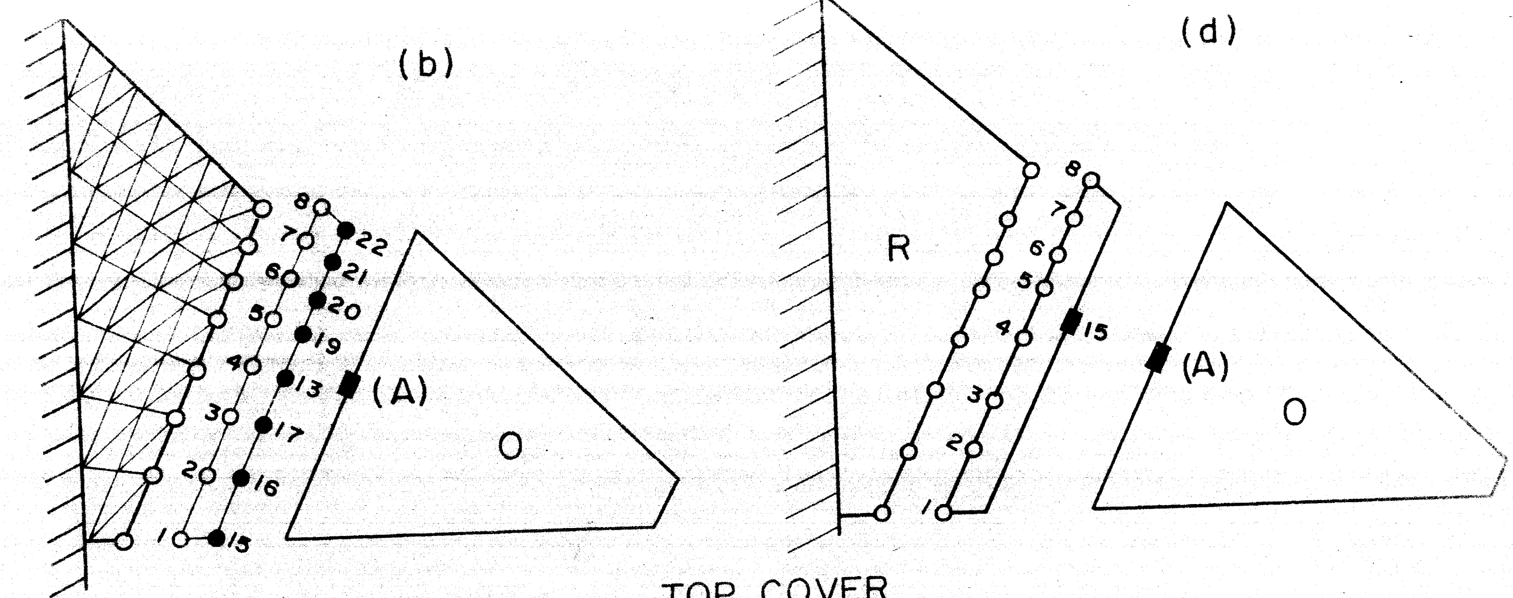


FIG. 3 COORDINATE SYSTEM FOR THIN WALLED BEAM THEORY

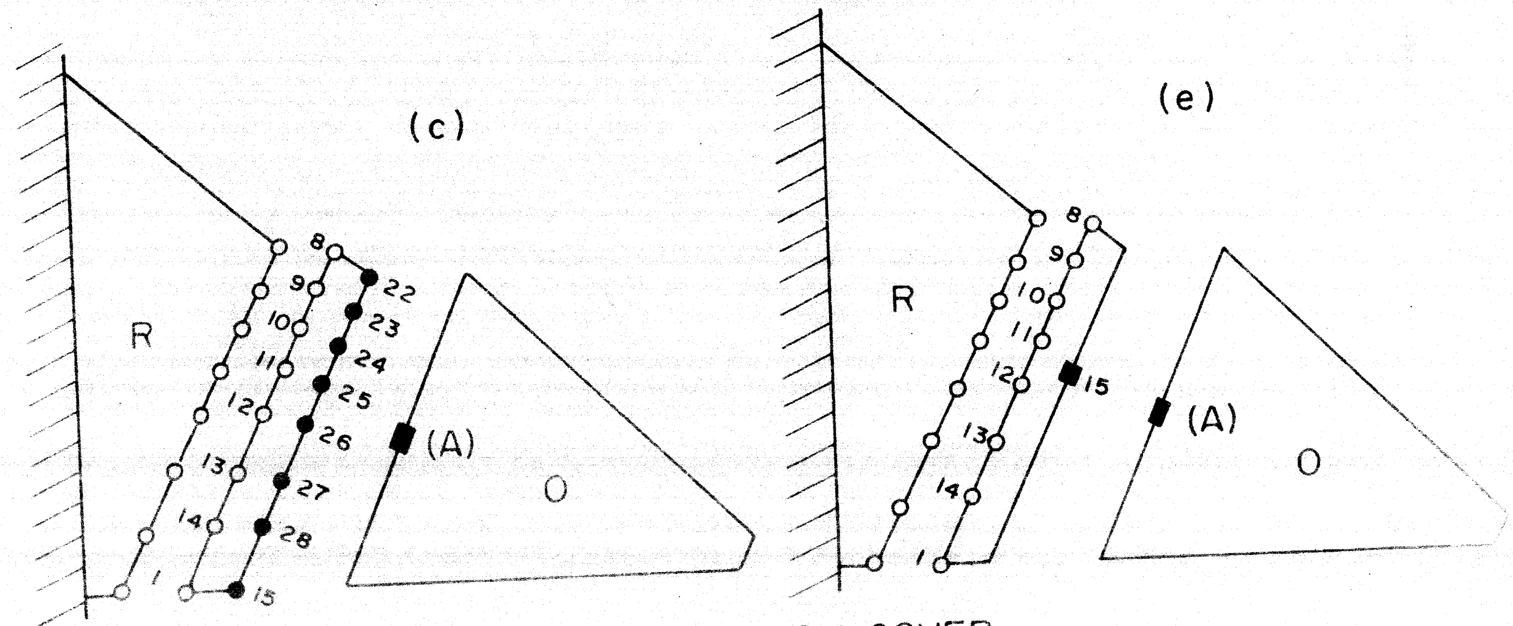


(a) FINITE ELEMENT IDEALISATION OF WING

FIG. 4



TOP COVER



BOTTOM COVER

FIG. 4 CONTD DETAILS OF JUNCTION ELEMENT

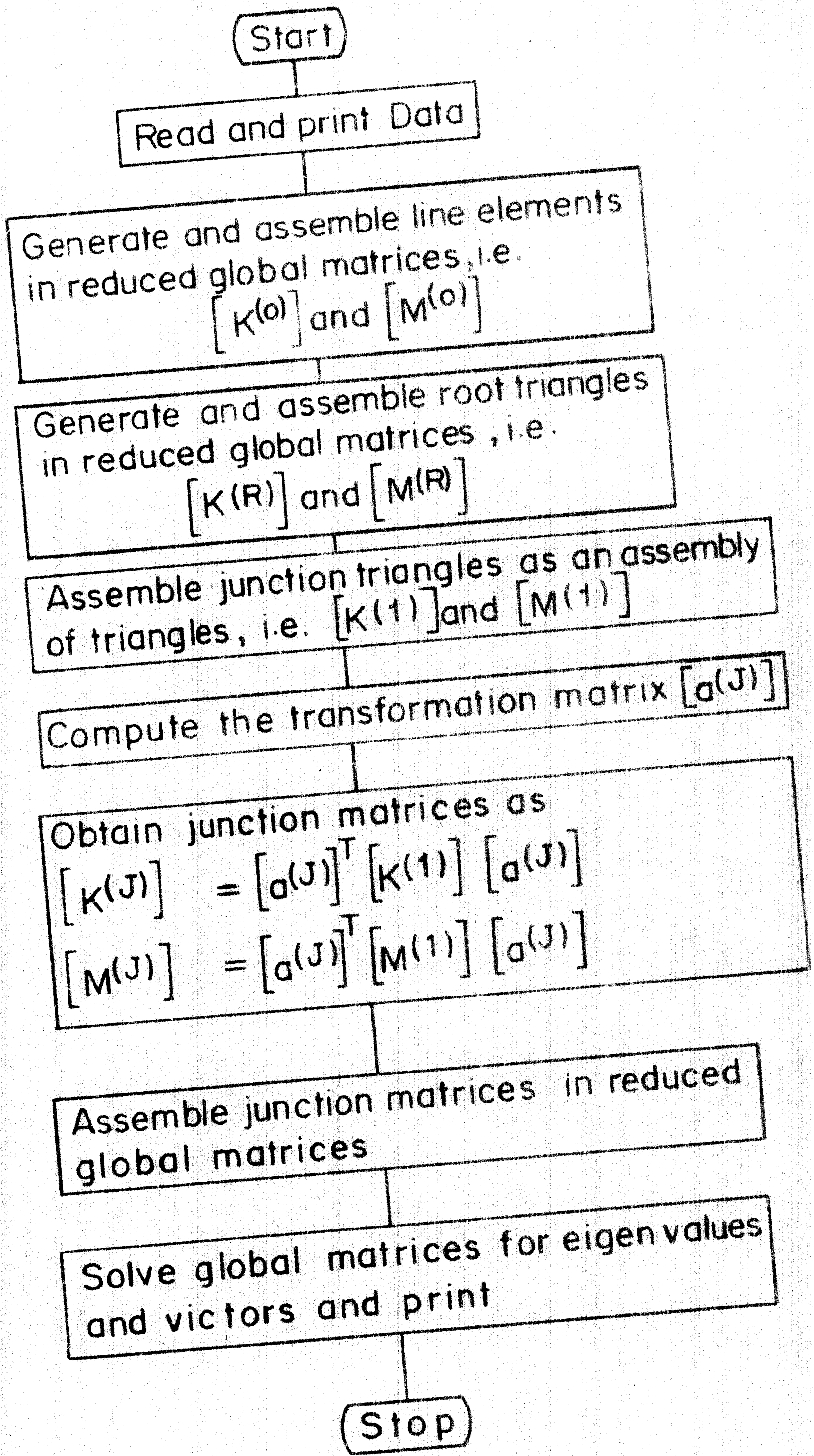
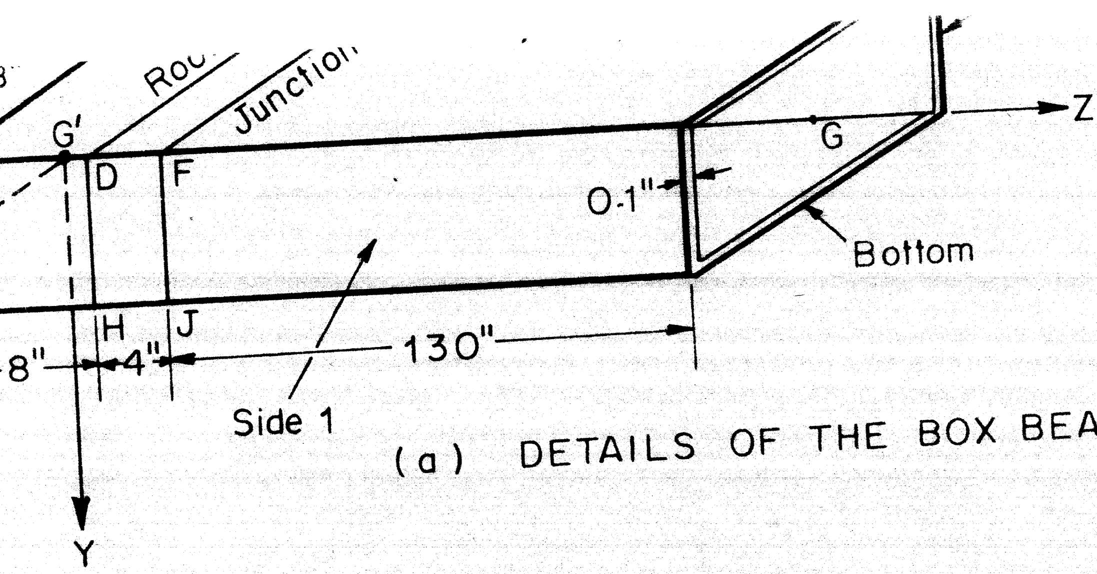
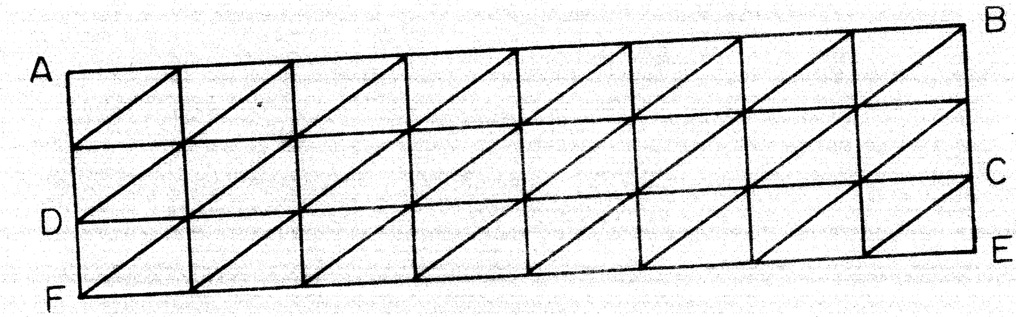


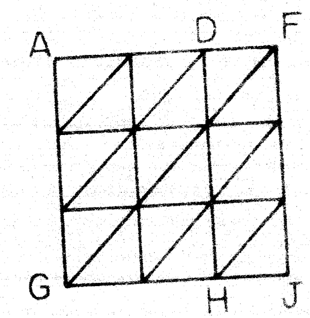
FIG. 5. SIMPLE FLOW CHART OF DRIVER PROGRAM



(a) DETAILS OF THE BOX BEAM



(b)



(c)

TRIANGULATION OF ROOT AND JUNCTION REGIONS
(b) TOP AND BOTTOM FACES (C) SIDES 1 AND 2

FIG. 6. FINITE ELEMENT IDEALISATION OF BOX-BEAM

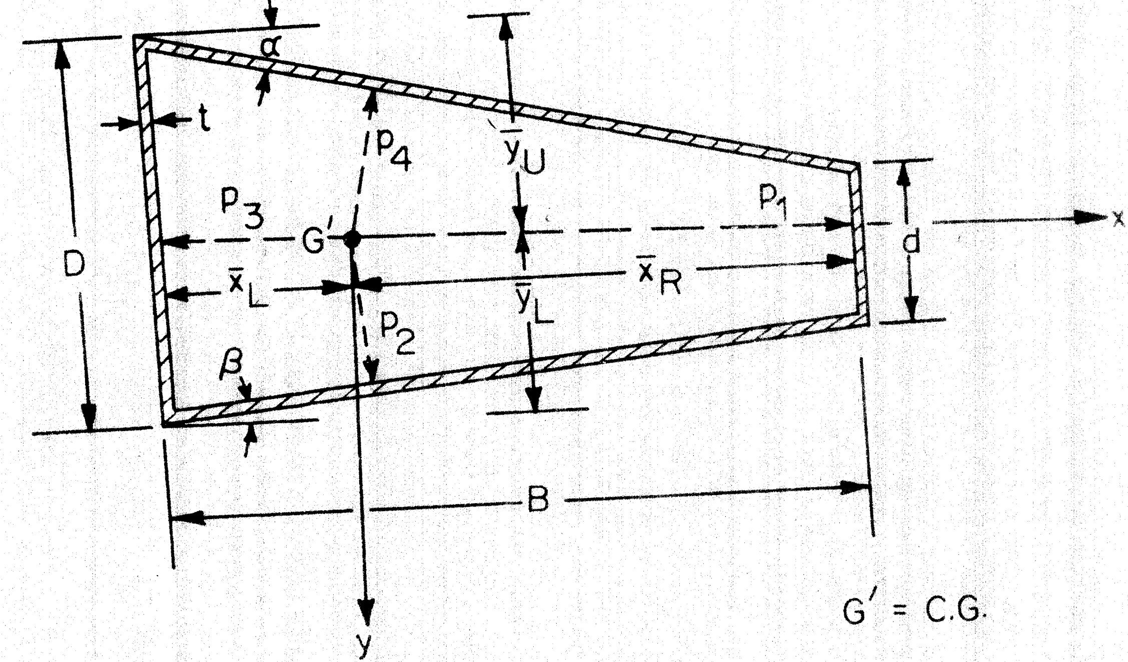


FIG. 7. DETAILS OF A TYPICAL CROSS-SECTION

APPENDIX - A

Matrices C_1 , C_2 and C_3 given in equations (57, 58 and 59)

A.1 Matrix C_1

$$[C_1] = \begin{bmatrix} [I_o]_{3x3} & [I_o]_{3x3} \\ [I_o]_{3x3} & [I_o]_{3x3} \end{bmatrix}$$

where $[I_o]$ is the symmetric matrix,

$$[I_o] = \begin{bmatrix} I_{xx} & \text{Symmetric} & \\ I_{xy} & I_{yy} & I_{y\bar{w}} \\ I_{x\bar{w}} & I_{y\bar{w}} & I_{\bar{w}\bar{w}} \end{bmatrix}$$

The expressions for elements of $[I_o]$ are given below.

$$[I_{xx}] = \oint_s x_2 t ds = t \Big[\bar{x}_R^2 (s_1 + s_2) + \bar{x}_L^2 (s_3 + s_4) -$$

$$\bar{x}_R \cos \beta . s_2^2 - \bar{x}_L \cos \alpha . s_4^2 + 1/3(\cos^2 \beta . s_2^3 +$$

$$\cos^2 \alpha \; s_4^3) \Big]$$

$$I_{xy} = \oint_s xy \; tds = t \Big[(y_1 \bar{x}_R . s_1 + y_2 \bar{x}_R . s_2 - \bar{y}_L . x_L \; s_3 + \bar{y}_U \bar{x}_L s_4)$$

$$+ 1/2 \Big\{ (\bar{x}_R s_1^2 + (\bar{x}_R \sin \beta - y_2 \cos \beta) \; s_2^2 + \bar{x}_L \; s_3^2 -$$

$$(\bar{x}_L \sin \alpha + \bar{y}_U \cos \alpha) \; s_4^2 \Big\} + 1/3 \Big\{ - \sin \beta \cos \beta \; s_2^3 +$$

$$\sin \alpha \cos \alpha \; s_4^3 \Big\} \Big]$$

$$I_{yy} = \oint_s y^2 \; tds = t \Big[(y_1^2 \; s_1 + y_2^2 \; s_2 + \bar{y}_L^2 \; s_3 + \bar{y}_U^2 \; s_4)$$

$$+ \Big\{ y_1 s_1^2 + y_2 \sin \beta \; . \; s_2^2 - \bar{y}_U \sin \alpha \; . \; s_4^2 \Big\}$$

$$+ 1/3 \; s_1^3 + \sin^2 \beta . s_2^3 + s_3^3 + \sin^2 \alpha . s_4^3 \Big\} \Big]$$

$$I_{x\bar{w}} = \oint_s x\bar{w} \; tds = t \Big[(\bar{x}_R \bar{w}_1 s_1 + \bar{x}_R \bar{w}_2 s_2 - \bar{x}_L \bar{w}_3 s_3 - \bar{w}_4 \bar{x}_L s_4)$$

$$+ 1/2 \Big\{ \bar{x}_R k' s_1^2 + (- \bar{w}_2 \cos \beta . + k'' \; \bar{x}_R) \; s_2^2 - \bar{x}_L k''' . s_3^2$$

$$+ (\bar{w}_4 \cos \alpha - k^{IV} \; \bar{x}_L) \Big\} \quad + 1/3 \Big\{ - k'' \cos \beta . s_2^3$$

$$+ k^{IV} \cos \alpha \; s_4^3 \Big\} \Big]$$

$$I_{y\bar{w}} = \oint_s y\bar{w}\ t ds = t \Big[(y_1 \bar{w}_1 s_1 + y_2 \bar{w}_2 s_2 + \bar{y}_L \bar{w}_3 s_3$$

$$- \bar{y}_U \bar{w}_4 s_4) + 1/2 \Big\{ (\bar{w}_1 + y_1 k') s_1^2 + (\bar{w}_2 \sin \beta + y_2 k'') s_2^2$$

$$+ (k''' \bar{y}_L - \bar{w}_3) s_3^2 + (\bar{w}_4 \sin \alpha - \bar{y}_U k^{IV}) s_4^2 \Big\}$$

$$+ 1/3 \Big\{ k' s_1^3 + k'' \sin \beta\ s_2^3 - k''' s_3^3 + k^{IV} \sin \alpha . s_4^3 \Big\} \Big]$$

$$I_{\bar{w}\bar{w}} = \oint_s \bar{w}^2 t ds = t \Big[(\bar{w}_1^2 s_1 + \bar{w}_2^2 s_2 + \bar{w}_3^2 s_3 + \bar{w}_4^2 s_4)$$

$$+ \Big\{ k' \bar{w}_1 s_1^2 + k'' \bar{w}_2 s_2^2 + k''' \bar{w}_3 s_3^2 + k^{IV} \bar{w}_4 s_4^2 \Big\}$$

$$+ 1/3 \quad k'^2 s_1^2 + k''^2 s_2^2 + k''' s_3^2 + k^{IV^2} s_4^2 \Big\} \Big]$$

A.2. Matrix $[C_2]$

$$[C_2] = \begin{bmatrix} [0]_{2x2} & [0]_{2x4} \\ [0]_{4x2} & [S]_{4x4} \end{bmatrix}$$

where $[0]$ indicates Null-matrix and $[S]$ is the symmetric matrix whose elements are given below.

$$S_{11} = \oint_s \left(p - \frac{d\bar{w}}{ds}\right)^2 t ds = \left\{ (p_1 - k')^2 s_1 + (p_2 - k'')^2 s_2 + (p_3 - k''')^2 s_3 + (p_4 - k^{IV})^2 s_4 \right\} t$$

$$S_{12} = - \oint_s \left(p - \frac{d\bar{w}}{ds}\right)\left(\frac{dx}{ds}\right) t ds = - t \left\{ (p_2 - k'') . \cos \beta . s_2 + (p_4 - k^{IV}) \cos \alpha . s_4 \right\}$$

$$S_{13} = - \oint_s \left(p - \frac{d\bar{w}}{ds}\right)\left(\frac{dy}{ds}\right) t ds = - t \left\{ (p_1 - k') s_1 + (p_2 - k'') \sin \beta . s_2 + (p_3 - k''') s_3 + (p_4 - k^{IV}) \sin \alpha . s_4 \right\}$$

$$S_{14} = - \oint_s \left(p - \frac{d\bar{w}}{ds}\right)\left(\frac{d\bar{w}}{ds}\right) t ds = - t \left\{ (p_1 - k') k' s_1 + (p_2 - k'') k'' s_2 + (p_3 - k''') s_3 + (p_4 - k^{IV}) k^{IV} s_4 \right\}$$

$$S_{22} = \oint_s \left(\frac{dx}{ds}\right)^2 t ds = t \left\{ (\cos^2 \beta . s_2 + \cos^2 \alpha . s_4) \right\}$$

$$S_{23} = \oint_s \frac{dx}{ds} . \frac{dy}{ds} t ds = t (- \sin \beta . \cos \beta \ s_2 + \sin \alpha . \cos \alpha . s$$

$$k'' \cos \beta . \ s_2 + k^{IV} \cos \alpha . s_4)$$

$$S_{33} = \oint_s \left(\frac{dy}{ds}\right)^2 tds \qquad = t\,(s_1 + \sin^2\beta . s_2 + s_3 + \sin^2\alpha . s_4)$$

$$S_{34} = \oint_s \frac{dy}{ds} \cdot \frac{d\bar{w}}{ds}\, tds \qquad = t\,(k' s_1 + k'' \sin\beta\; s_2 - k''' s_3$$
$$+ k^{IV} \sin\alpha\; s_4)$$

$$S_{44} = \oint_s \left(\frac{d\bar{w}}{ds}\right)^2 tds \qquad = t\,(k'^2 s_1 + k''^2 s_2 + k'''^2 s_3$$
$$+ k^{IV^2} s_4)$$

A.3. The Matrix $[C_3]$

$$[C_3] = \begin{bmatrix} [H]_{3x3} & [0]_{3x3} \\ [0]_{3x3} & [0]_{3x3} \end{bmatrix}$$

The elements of H matrix are given under:

$$H_{11} = H_{22} = \oint_s tds = (2B + D + d)\; t$$

$$H_{12} = 0$$

$$H_{13} = - \oint_s y\; tds = -\, t\,\{ y_1 s_1 + y_2 s_2 + \bar{y}_L s_3 - \bar{y}_U s_4$$

$$- 1/2\,(s_1^2 - s_3^2 + s_2^2 \sin\beta + s_4^2 \sin\alpha)\}$$

k', k'', k''' and k^{IV} : Constants used in warp expressions (given in Appendix - B).

$$y_1 = (\tan \alpha - \bar{y}_U)$$

$$y_2 = \bar{y}_L - B \tan \beta$$

p_1, p_2, p_3 and p_4 : Length of the perpendicular from the origin to the side of the cross-section (for expressions see Appendix - B).

$\bar{x}_L, \bar{x}_R, \bar{y}_U$ and $\bar{y}_L$: Distance of centre of gravity (origin) from the sides of the cross-section (see Figure 7 and Appendix - B).

$$\bar{w}_2 = \bar{w}_1 + k'd$$

$$\bar{w}_3 = \bar{w}_2 + \frac{k''B}{\cos \beta}$$

$$\bar{w}_4 = \bar{w}_3 + k''' D$$

where

$$k' = k(\frac{1}{\delta'} - \frac{p_1}{2A})$$

$$k'' = k(\frac{1}{\delta'} - \frac{p_2}{2A})$$

$$k''' = k(\frac{1}{\delta'} - \frac{p_3}{2A})$$

$$k^{IV} = k^{IV}(\frac{1}{\delta'} - \frac{p_4}{2A})$$

$$\text{with } k = \frac{(D + d + B/\cos \alpha + \bar{B}/\cos \beta)}{2AGt}$$

$$p_1 = \bar{x}_R$$

$$p_2 = \bar{y}_L \cos \alpha - (B-\bar{x}_R) \sin \alpha$$

$$p_3 = \bar{x}_L$$

$$p_4 = (D-\bar{y}_L) \cos \alpha - (B - \bar{x}_R) \sin \alpha$$

$$\bar{x}_R = \frac{B(D + 2d)}{3(D+d)}$$

$$\bar{x}_L = B - \bar{x}_R$$

$$_T = 2$$

$$_L = 2\left\{\frac{\frac{B \tan \beta}{3} + d(B \tan \beta + d/2) + \frac{B \tan \alpha}{2}(B \tan \beta + d + \frac{B \tan \alpha}{3}}{(B \tan \alpha + B \tan \beta\ 2d)}\right\}$$

$$\delta' = (D + d + B/\cos \alpha + B/\cos \beta)$$

$$A = \frac{B}{2}(D + d).$$

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K/vbg*